UNIVERSITY OF CAMBRIDGE Faculty of Education

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Mathematical Knowledge in Teaching

Seminar Series

Conceptualising and theorising mathematical knowledge in teaching (11-12 January 2007 - 2 days - Cambridge)

Professor Kenneth Ruthven & Dr Tim Rowland, University of Cambridge

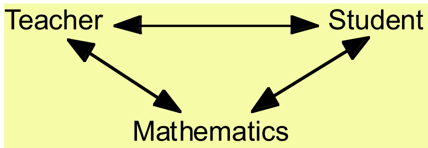
Changed Views on Mathematical Knowledge in the Course of Didactical Theory Development – Independent Corpus of Scientific Knowledge or Result of Social Constructions?

Heinz Steinbring, Universität Duisburg-Essen

Contents

1. Introduction: The didactical triangle

A **normative / prescriptive** or a **descriptive / analytic** schema?



Three questions of analyses:

Which explicit and implicit (unconscious) concepts and role descriptions exist about …

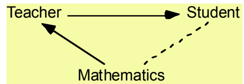
(A) … the three ›elements‹: mathematics, teacher and students?

(B) … the relationships or interactions between the three ›elements‹: mathematics, teacher and students?

(C) What is seen as the central and crucial means (among the three ›elements‹) of positively influencing and improving the learning process?

2. The ›stoffdidaktic‹ elaboration of mathematical knowledge

The didactical triangle as a **normative / prescriptive** schema!

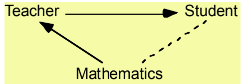


›*stoffdidaktik*‹ – more specifically – the ›*didactically oriented content analysis*‹:

»The research complex of didactically oriented content analysis (Sachanalysen) has lately engaged mathematics education in the Federal Republic of Germany in a particular way. ... The research methods of this area are identical with those of mathematics, … ... The goal of the ›didactically oriented content analysis‹ which essentially work with mathematical methods is giving a better foundation for the formulation of the content related learning goals and for the development, the definition and the use of a differentiated methodical set of instruments.« (Griesel 1974, p. 118).

2. The ›stoffdidaktic‹ elaboration of mathematical knowledge

The didactical triangle as a **normative / prescriptive** schema!



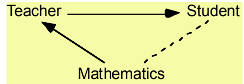
›*stoffdidaktik*‹ – the ›*didactically oriented content analysis*‹:
a prototype for stoffdidaktic:

- uniform mathematics as elaborated by Bourbaki
- the ›New Mathematics‹ movement

The stoffdidaktic analysis of school-mathematics of higher school grades as well as of primary school put **uniform**, **logically consistent scientific mathematics** (**new contents** and more **rigorous** concepts) into the centre of their work.

2. The ›stoffdidaktic‹ elaboration of mathematical knowledge

The didactical triangle as a **normative / prescriptive** schema!

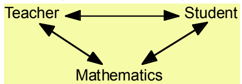


Three questions – (A), (B) & (C)

- The mathematical content – a uniform, objective and unchangeable content of teaching and learning according to the paradigm of scientific mathematics.
- Teaching, learning and understanding processes – are based on the rigid subject matter structures; the teacher is the ›giver‹ – the student the recipient – of the didactically prepared content.
- The scientific stoffdidactic elaboration of mathematical knowledge is the central means of steering and optimizing the mathematical instruction, learning and understanding processes.

3. Dynamics of knowledge development and processes of teaching and learning

The didactical triangle as a **normative / prescriptive** schema? First changes!

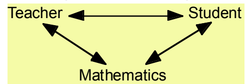


Heinrich Winter: »So-called *Sachanalysen* can have a downright calamitous effect on the school reality, if these refer reductionistically solely to mathematics (perhaps even to assumed mathematics) and fade out other essential constituents of learning mathematics.

- [One] inevitably encounters problems of the goals and forms of learning itself, which are not or hardly explained in the *Sachanalysen*.
- In general: *Sachanalysen* are in danger of losing focus on the outer-mathematical reality and thus on the students' experience of the world, and this is only one pedagogical sin of such reductions« (Winter 1985, p. 80/81).

3. Dynamics of knowledge development and processes of teaching and learning

The didactical triangle as a **normative / prescriptive** schema ? First changes!



Changed views on mathematical knowledge and on learning:

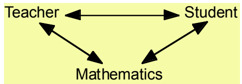
Freudenthal – Mathematics as *process* instead of *product*:

»Every mathematician knows at least unconsciously that besides ready-made mathematics there exists mathematics as an activity. … The opposite of ready-made mathematics is mathematics *in statu nascendi*. … the pupil himself should re-invent mathematics. … The learning process has to include phases of directed invention, that is, of invention not in the objective but in the subjective sense, seen from the perspective of the student« (Freudenthal 1973, p. 114 / 118).

Mathematical Knowledge in Didactical Theory Development

3. Dynamics of knowledge development and processes of teaching and learning

The didactical triangle as a **normative / prescriptive** schema ? First changes!



Changed views on mathematical knowledge and on learning:

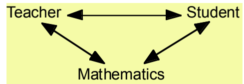
Wittmann – MATHEMATICS as general, cultural & historical knowledge:

»… mathematics educators need a lively interaction with MATHEMATICS, and they must devote an essential part of their professional life to stimulating, observing, and analyzing genuine MATHEMATICAL activities of children, students and student teachers. Organizing and observing the fascinating encounter of human beings with MATHEMATICS is the very heart of didactic expertise and forms a natural context for professional exchange with teachers« (Wittmann 1995, p. 358/9).

Mathematical Knowledge in Didactical Theory Development

3. Dynamics of knowledge development and processes of teaching and learning

The didactical triangle as a **normative / prescriptive** schema ? First changes!



Changed views on mathematical knowledge and on learning:

Jahnke & Otte – Mathematics as *theoretical* knowledge:

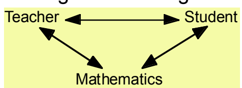
»For didactics, for instance, it is obvious that the didactic problem in its deeper sense, that is in the sense that it is necessary to work on it scientifically, is constituted by the very fact that **concepts will reflect relationships, and not things**« (Jahnke & Otte 1981, pp. 77/78).

Such a view – that mathematical knowledge does not reflect things but relations – implies a refined understanding of teaching / learning mathematics focusing on autonomous activities of the participating persons; these persons have to construct the conceptual relations!

Mathematical Knowledge in Didactical Theory Development

3. Dynamics of knowledge development and processes of teaching and learning

The didactical triangle as a **normative / prescriptive** schema ? First changes!



Changed views on mathematical knowledge and on learning:

IDM research group »Mathematics Teacher Education«:

»… ›teaching‹ cannot be derived from the descriptions of ›learning‹ … The specific of teaching lies within the content of the activity, which aims at effectuating learning«.

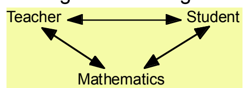
Bromme: central aspects of the teacher activity (for example the preparation of mathematics instruction) under the perspective: **teachers as experts of their profession**.

Steinbring: ›**epistemological knowledge for mathematics teachers**‹ with a view to the *theoretical* and *dynamic* character of mathematics.

Mathematical Knowledge in Didactical Theory Development

3. Dynamics of knowledge development and processes of teaching and learning

The didactical triangle as a **normative / prescriptive** schema ? First changes!



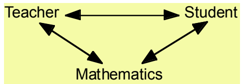
Three questions – (A), (B) & (C)

- The three elements of the didactical triangle detach from each other and gain more independence and own dynamics together with new didactical research questions.
- The relations between these elements are of indirect nature:
  – the teacher is seen like a moderator of learning processes,
  – the student is responsible for his mathematical learning processes.
- The ›functioning‹ of the didactical triangle represents a reciprocal process between its three elements and it is not a linear nor circular transport of mathematics via the teacher to the students etc .

Mathematical Knowledge in Didactical Theory Development

4. Education research and mathematical teaching-learning-practice

The didactical triangle as a **descriptive / analytic** schema!



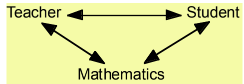
The *particular nature* of the relation between *theory* and *practice*?

- (Implicit) assertion: Mathematics educational research and developmental work can always bring about support, positive influences and direct improvements for mathematical teaching and learning processes in school (and university).
- This long lasting, traditional assumption, that within educational research knowledge and contents of subject matter are carefully elaborated and then directly transferred into school practice, has been decidedly criticized and replaced by other conceptions.

Mathematical Knowledge in Didactical Theory Development

4. Education research and mathematical teaching-learning-practice

The didactical triangle as a **descriptive / analytic** schema!



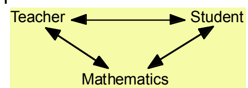
IDM research group (around H. Bauersfeld): An essential criticism on this *theory-practice*-position (connected with ›stoffdidaktik‹)

»… the ›*turn to everyday life*‹ … with its criticism on ›holiday didactics‹ … contained the claim of assigning a greater meaning than before to the features of everyday instruction. In ethnographic instruction observations and interpretative studies, one saw a corrective for instruction conceptions, which emerge at the didactical desk; one was disillusioned by the effects of the school reforms … and wanted to understand better the surprising stability of everyday instruction, its own progress and its traditions« (Voigt 1996, p. 384).

Mathematical Knowledge in Didactical Theory Development

4. Education research and mathematical teaching-learning-practice

The didactical triangle as a **descriptive / analytic** schema!



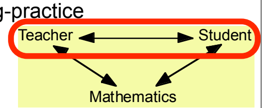
IDM research group (around H. Bauersfeld): An essential criticism on this *theory-practice*-position (connected with ›stoffdidaktik‹)

- (School) *Practice* & (mathematics education) *Theory* need to be seen as two relatively autonomous institutions and fields of work.
- Each of the two fields is subject to its own expectations and aims, as well as to system-internal requirements and norms, which cannot be invalidated from the outside in order to apparently be able to directly interfere into and to purposefully regulate within the other field.

Mathematical Knowledge in Didactical Theory Development

4. Education research and mathematical teaching-learning-practice

The didactical triangle as a **descriptive / analytic** schema!



IDM research group (around H. Bauersfeld): An essential criticism on this *theory-practice*-position (connected with ›stoffdidaktik‹)

The interactionist perspective (Bauersfeld et al.) relies mainly on two (until then neglected) basic aspects: the *learning child* (in the classroom) and the *interaction between the learner and the teacher*.

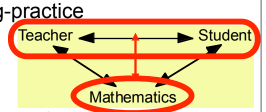
two perspectives:

- »The one is an individual-psychological perspective which emphasizes the learner‘s autonomy and his cognitive development …
- The other is a collectivistic perspective which … understands learning mathematics as the socialization of the child into a given classroom culture ...« (Voigt 1994, p.78).

Mathematical Knowledge in Didactical Theory Development

4. Education research and mathematical teaching-learning-practice

The didactical triangle as a **descriptive / analytic** schema!



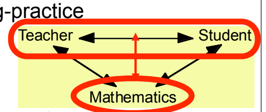
Steinbring (2005): interaction-research approach of the *social epistemology of mathematical knowledge*

- the particularity of the social existence of mathematical knowledge is an essential component of this research approach of interaction analysis
- the subject matter ›mathematics‹ – according to the positions developed in the previous parts – is not understood as a pre-given, finished product, but interpreted according to the epistemological conditions of its dynamic, interactive development

Mathematical Knowledge in Didactical Theory Development

4. Education research and mathematical teaching-learning-practice

The didactical triangle as a **descriptive / analytic** schema!



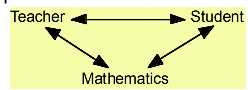
Steinbring (2005): interaction-research approach of the *social epistemology of mathematical knowledge*

- Mathematical concepts are *constructed* in interaction processes as symbolic relational structures and are coded by means of *signs and symbols*, that can be combined logically / consistently in mathematical operations.
- This interpretation of mathematical knowledge does not require a description for the mathematical knowledge as completely fixed and pre-given – the symbolic relations have to be actively constructed and controlled by the subjects in interactions.
- More details of the *social epistemology of mathematical knowledge* in part 5.

Mathematical Knowledge in Didactical Theory Development

4. Education research and mathematical teaching-learning-practice

The didactical triangle as a **descriptive / analytic** schema!



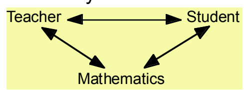
Three questions – (A), (B) & (C)

- The three elements are seen as relatively independent ›systems‹ within their institutional contexts and they are engaged in reciprocal interactions with each other.
- The teacher-students interactions take place between autonomous subjects, who are aware of each other in communication, but who cannot directly influence the consciousness of the other.
- Changes and improvements cannot be forced from the outside or by direct intervention. Changes can only be stimulated in the participating autonomous systems and then they need to be continued and realized within the systems themselves.

Mathematical Knowledge in Didactical Theory Development

5. Mathematical knowledge development in interaction – A case study

Interactive construction of arithmetical features of the elementary number concept.



- Mathematics teaching is understood as an ***autonomous culture***, that is not directly determined by mathematics as a scientific discipline.
- It is a culture with a specific type of (school-) mathematical knowledge, using **mathematical symbols** and a **particular mathematical discourse**.

»Without a symbolic apparatus to convey our ideas to one another, and to pass on our results to future generations, there wouldn't be any such thing as mathematics – indeed, there would be essentially no culture at all ….« (Wilder 1986, p. 193).

Mathematical Knowledge in Didactical Theory Development

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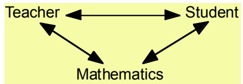

Mathematics learning and teaching as a complex interaction process negotiating and constructing specific meanings for **mathematical signs** and **symbols**.

Two functions of the role of mathematical signs:

(1) A **semiotic** function: the role of the mathematical sign as »something which stands for something else«.

(2) An **epistemological** function: the role of the mathematical sign in the frame of the epistemological constraints of mathematical knowledge.

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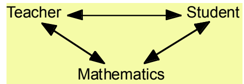
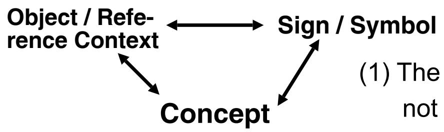


Interactive construction of meanings for **mathematical signs** & **symbols**

The Epistemological Triangle



(1) The mediation ›Object – Sign‹ is not between concrete things or properties but between **relational structures**.

(2) The mediation ›Object – Sign‹ is regulated by some pre-supposed **mathematical conceptual ideas** and vice versa the unfolding mediation will produce **new conceptual mathematical knowledge**. (Steinbring 2005)

Mathematical Knowledge in Didactical Theory Development

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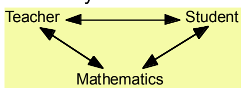
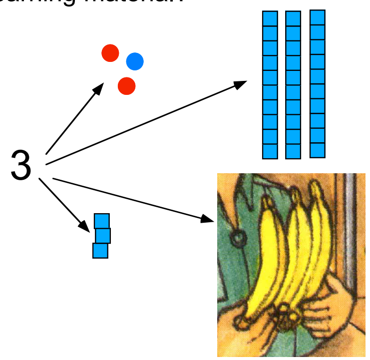
Interactive construction of arithmetical features of the elementary number concept.



Interactive construction of meanings for **mathematical signs** & **symbols**

Are the numbers hidden in the learning material?

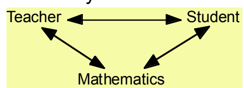
What **is** ›Three‹?



Mathematical Knowledge in Didactical Theory Development

5. Mathematical knowledge development in interaction – A case study

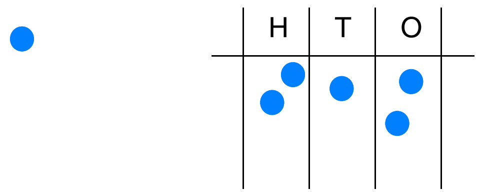
Interactive construction of arithmetical features of the elementary number concept.



Interactive construction of meanings for **mathematical signs** & **symbols**

A young child: Everything **can be** ›Three‹!

A single chip is not always 1!



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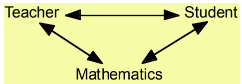
Interactive construction of arithmetical features of the elementary number concept.



Interactive construction of meanings for **mathematical signs** & **symbols**

Paul Benacerraf:

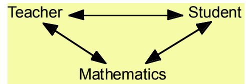
Numbers are neither objects nor names for objects!

»I therefore argue, … that numbers could not be objects at all; for there is no reason to identify any individual number with any one particular object than with any other (not already known to be a number)« (Benacerraf 1984, p. 290/1).

But if numbers are not objects, what else are they?

5. Mathematical knowledge development in interaction – A case study

Interactive construction of arithmetical features of the elementary number concept.



Interactive construction of meanings for **mathematical signs** & **symbols**

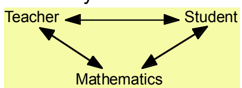
Paul Benacerraf:

But if numbers are not objects, what else are they?

»To be the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5…… Any object can play the role of 3; that is any object can be the third element in some progression. …« (Benacerraf 1984, p. 291)

5. Mathematical knowledge development in interaction – A case study

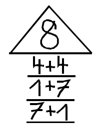
Interactive construction of arithmetical features of the elementary number concept.



Interactive construction of meanings for **mathematical signs** & **symbols**

- Mathematical knowledge does not relate directly to concrete or real objects. Mathematical concepts coded in signs and symbols represents abstract **relationships**, **structures** and **patterns**.
- Mathematical knowledge is **theoretical knowledge** and cannot be reduced to empirical objects or concrete properties.

KLAUS and sönke work on number houses – Example of a mathematical interaction

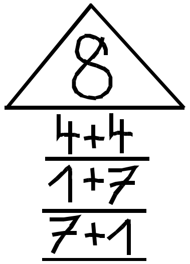


The teaching episode is taken from a mathematics lesson in the frame of the flexible entrance phase into school with children of different age (grades 1 & 2).

The following document was videotaped in the research project: »**M**athematics teaching in multi-**a**ged **l**earning groups 1 and 2 – **In**teraction and interventio**n** (Malin)«, (Dr. M. Nührenbörger, Universität Duisburg-Essen).

Mathematical Knowledge in Didactical Theory Development

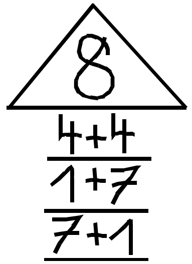
KLAUS and sönke work on number houses –
Example of a mathematical interaction



KLAUS (the older) and sönke (the younger) work together on number house ›8‹. They have already found the decompositions ›4 + 4‹ (written down by KLAUS), ›1 + 7‹ (written down by sönke) and ›7 + 1‹ (written down by KLAUS).

Mathematical Knowledge in Didactical Theory Development

KLAUS and sönke work on number houses –
Example of a mathematical interaction



Phase 1.1 [1 - 12]

KLAUS explains the mathematical notation of the number Six.

Mathematical Knowledge in Didactical Theory Development

KLAUS and sönke work on number houses –
Example of a mathematical interaction

d+2

Aspects of analysis Phase 1.1

KLAUS and sönke negotiate the meaning of the sign d .

In which manner the number Six is usually written as a mathematical sign?

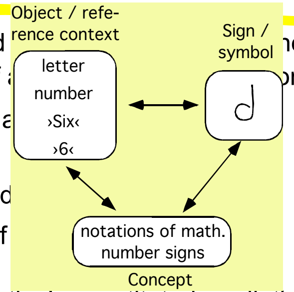
The sign d questioned by KLAUS is interpreted by using different referential attributes – it could be a letter (a d) or a number (the number 6).

Mathematical Knowledge in Didactical Theory Development

KLAUS and sönke work on number houses –
Example of a mathematical interaction

d+2

Aspects of analysis Phase 1.1



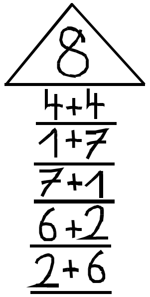
KLAUS and sönke negotiate the meaning of the sign d .

The interactively constituted mediation between ›sign / symbol‹ and ›object / reference context‹ is *conventionally* determined and is not inter related by means of *conceptual, mathematical* aspects.

Not (yet) a true epistemological – but conventional – connection between sign and referent.

Mathematical Knowledge in Didactical Theory Development

KLAUS and sönke work on number houses –
Example of a mathematical interaction



Phase 1.2 [13 - 36]

sönke finds an exchange task to 4+4

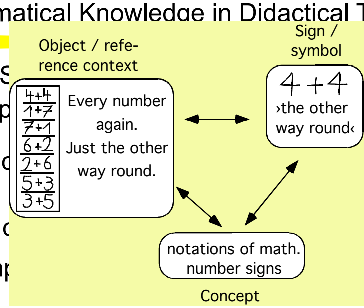
Mathematical Knowledge in Didactical Theory Development

KLAUS ... uses –
Examp

Aspec

sönke c

decomp



4+4 4+4

sönke: »… Four plus four is then also the other way round, just you can't see it .« (34).

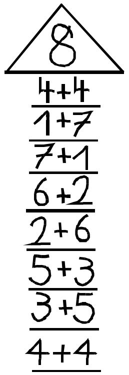
This four plus four is differently written, this way: 4+4

sönke's justification:

☞ All decompositions appear twice, therefore also 4 + 4 two times.
Klaus accepts (but is this also a *mathematical* explanation?)

Mathematical Knowledge in Didactical Theory Development

KLAUS and sönke work on number houses –
Example of a mathematical interaction



Phase 2.1 [71 - 92]

The teacher makes aware of a doubling of the task 4 + 4 and of missing decompositions

Mathematical Knowledge in Didactical Theory Development

KLAUS and sönke work on number houses –
Example of a mathematical interaction

4+4 4+4

Aspects of analysis Phase 2.1

Teacher: »… **double** tasks … take these out.« (81)

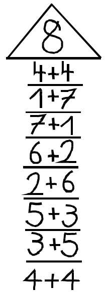
KLAUS: compares with 6 + 2 and 2 + 6

Teacher: also an **exchange task**? 4+4

Teacher: the same numbers, but in a **different sequence** (6 + 2 and 2 + 6)

Teacher: these two one cannot distinguish:

4+4 4+4



Mathematical Knowledge in Didactical Theory Development

KLA… ses –

Exa…

Asp…

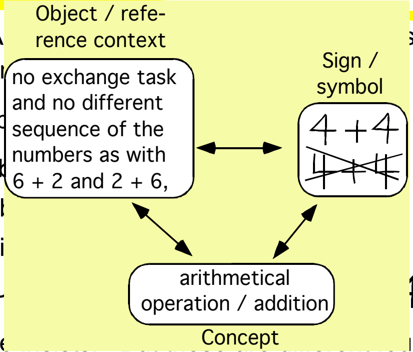
doub…

numb…

be di…

KLAU… 4

sönke … rs«

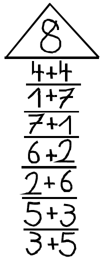


4+4 4+4

The teacher justifies the »sameness« of both decompositions with ›didactical‹ characterisations (cf. exchange task).

4+4

4+4



Mathematical Knowledge in Didactical Theory Development

KLAUS and sönke work on number houses –
Example of a mathematical interaction

4+4 4+4

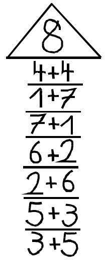
Aspects of analysis Phase 2.1

Do the ›didactical‹ characterisations offer in the course of interaction epistemologically acceptable *mathematical* knowledge constructions?

Is the observable different notation of the number signs an acceptable *mathematical* justification?

4

4

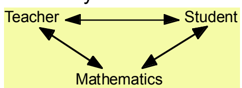


Are distinguishable concrete actions with (learning-) material an acceptable *mathematical* justification?

Mathematical Knowledge in Didactical Theory Development

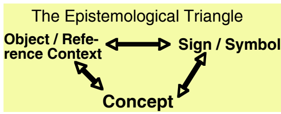
5. Mathematical knowledge development in interaction – A case study

Theorisation of classroom knowledge building –
Mathematical knowledge constructed in interaction.



The analysis of this episode demonstrates a new interpretation from a mathematics education research perspective:

Mathematical knowledge is interactively constructed by the participants on the basis of specific *epistemological* constraints. Such epistemological conditions are effective also within instructional learning processes, and there they bring about a *socially generated epistemology* of school-mathematical knowledge.



Object / Reference Context
<—> Sign / Symbol

conventional
methodical
didactical
theoretical